Lecture 5

Physics 404

We start our study of statistical physics with a very simple "toy model" of N spins on a 1D lattice. Each spin sits on a lattice site and does not move. It does not interact with any of its neighbors. It can only interact with an external magnetic field, having an energy $U = \pm m_e B$, where the - sign applies for an "up" spin aligned with the magnetic field, and + applies to "down" spins.

Now ask 3 questions about this toy model.

1) How many microscopic states are available to this system of N spins? If we imagine constructing this lattice of spins from scratch, one has two possible ways of installing the first spin (up or down), two more independent choices for the second spin, etc. Overall the number of ways is the product of all these independent possibilities, namely 2^N. If we turn on a uniform magnetic field over the entire spin system, the total energy of all the spins is $U = -\sum_{i=1}^{N} \vec{m}_i \cdot \vec{B}$. Since each spin sees the same magnetic field, this can be written in terms of a **macroscopic** quantity, the total magnetic moment $M = \sum_{i=1}^{N} m_i$, where $m_i$is the magnetic moment of the ith spin, giving $U = -MB$.
2) A second question now arises: How many **macroscopic** magnetic moment values *M* are possible for a lattice of N spins? By starting with all the spins aligned "up" (with $M = Nm_e$), and flipping one spin at a time, one can show that the total number of distinct moment values is N+1. Note that if we take N = 10, there are 11 possible values of M, but $2^{10} = 1024$ microscopic states available to the system. This difference grows rapidly with increasing N, showing that **there is a large <u>multiplicity</u> of microscopic states corresponding to each macroscopic state.**
3) This leads to a third question: What is the exact multiplicity of each macroscopic state of the system? In other words, how many micro-states of an N-spin lattice correspond to the same macroscopic magnetic moment M? Introduce some new notation: call the number of up spins $N_\uparrow$ and the number of down spins $N_\downarrow$. Define the "spin excess" 2s through $N_\uparrow = N/2 + s$ and $N_\downarrow = N/s - s$, such that $N_\uparrow - N_\downarrow = 2s$, with $N_\uparrow + N_\downarrow = N$. The total magnetic moment can now be written as $M = N_\uparrow m_e - N_\downarrow m_e = (2s)\, m_e$, hence *M* and *2s* are basically interchangeable quantities. The multiplicity can be calculated by starting with all the spins down, and deciding how many ways there are to create $N_\uparrow$ spins in that lattice, without regard to the order in which the spins are flipped. The answer is found to be the "binomial coefficient" $g(N, N_\uparrow, N_\downarrow) = \frac{N!}{N_\uparrow!\, N_\downarrow!}$, or $g(N,s) = \frac{N!}{\left(\frac{N}{2}+s\right)!\left(\frac{N}{2}-s\right)!}$.

The multiplicity function should count the total number of states properly. In other words, we expect the multiplicity function to satisfy $2^N = \sum_{s=-N/2}^{N/2} g(N,s)$. This is found to be the case, using the binomial expansion for the quantity $(x+y)^N$, and setting $x = y = 1$.

The multiplicity function is well approximated by a continuous Gaussian distribution function in the limit N>>1 and |s|<<N. Look at the figures on the first two pages of the accompanying pdf file entitled "Spin Multiplicity and the Gaussian Approximation". It has the form $g(N,s) \cong \sqrt{\frac{2}{\pi N}}\, 2^N e^{-2s^2/N}$. One

can show that this approximate version of the multiplicity function also correctly counts all of the micro-states by integrating over s as: $\int_{-N/2}^{N/2}\sqrt{\frac{2}{\pi N}}2^N e^{-2s^2/N}ds \cong \int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi N}}2^N e^{-2s^2/N}ds = 2^N$. The Gaussian integral is derived in Appendix A on page 439 of K+K.

The multiplicity distribution becomes more and more strongly peaked as the number of spins *N* increases. This is illustrated on the third page of the accompanying pdf file. A plot of *g(N,s)/g(N,0)* versus *s/N* shows that the Gaussian functions become more and more narrow with increasing N. This means that in the limit of large N, only a very small fraction of all the micro-states have a large and significant multiplicity. We shall see later that these states are the ones that dominate the thermodynamic properties of the system. One can find the 1/e half-width of the *g(N,s)/g(N,0)* distribution as, $s_e = \pm\sqrt{\frac{N}{2}}$. Calculating the fractional width of the distribution, one finds $\frac{s_e}{N} = \frac{1}{\sqrt{2N}}$, which diminishes with increasing *N*, as noted above. Imagine a macroscopic object with N ~ 10^{24} spins. The fractional width is $\frac{s_e}{N} \sim 10^{-12}$, which is extraordinarily narrow!